

ROOTS IN OPERATOR AND BANACH ALGEBRAS

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ABSTRACT. We show that several known facts concerning roots of matrices generalize to operator algebras and Banach algebras. We show for example that the so-called Newton, binomial, Visser, and Halley iterative methods converge to the root in Banach and operator algebras under various mild hypotheses. We also show that the ‘sign’ and ‘geometric mean’ of matrices generalize to Banach and operator algebras, and we investigate their properties. We also establish some other facts about roots in this setting.

In memoriam Charles Read—gentleman, brother, mathematical force of nature.

1. INTRODUCTION

An *operator algebra* is a closed subalgebra of $B(H)$, for a complex Hilbert space H . In this paper we show that several known facts concerning roots of matrices generalize to operator algebras and Banach algebras. We begin by establishing some basic properties of roots that do not seem to be in the literature (although they may be known to some experts), as well as reviewing some that are. We then show that the ‘sign’ of a matrix generalizes to Banach algebras, and that Drury’s variant of the ‘geometric mean’ of matrices generalizes to operators on a Hilbert space (we also generalize his definition slightly), and prove some basic facts about these. We also show that the so-called Newton (or Babylonian), binomial, Visser, and Halley iterative methods for the root converge to the root in Banach and operator algebras under various mild hypotheses inspired by the matrix theory literature. Some parts of our paper are fairly literal transfers of matrix results to the operator or Banach algebraic setting, using known tricks or standard theory, and here we will try to be brief. However we have not seen these in the literature and they seem quite useful. For example our results, particularly probably the geometric mean, should be applicable to our ongoing study of ‘real positivity’ in operator algebras (see e.g. [9, 10, 11, 8, 6] and references therein) initiated by the first author and Charles Read.

Turning to background and notation, it is common when studying roots to make the assumption that the spectrum contains no strictly negative numbers. Note that a singular matrix with no strictly negative eigenvalues, may not have a square root (for example, E_{12} in M_2), or may have a square root but not have a square root in $\{x\}''$ (for example, E_{12} in M_3 , which has many square roots including $E_{13} + E_{32}$), or may have infinitely many square roots in $\{x\}''$ (for example, 0 in an algebra with

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trivial product). However in a Banach algebra and for $p \in \mathbb{N}$, any element x of type M (defined below; this is also sometimes called being ‘sectorial’, see e.g. [19], but there is ambiguity in the literature), and also any element whose (closed) numerical range (defined below) contains no strictly negative numbers, has a unique p th root with spectrum in a sector $S_{\frac{\pi}{p}}$ (see [29, 26] and also Theorem 2.5 below), and this root is in the closed subalgebra generated by x , which in turn is a subset of the second commutant $\{x\}''$. Here S_{θ} is the set of complex numbers with argument in $[-\theta, \theta]$. Thus we will usually (but not always) take roots of elements with no strictly negative numbers in its numerical range. Indeed sometimes we will require the numerical range to be in S_{θ} for some $\theta < \pi$.

A *unital* Banach algebra has an identity of norm 1. The *states* of A are the norm 1 functionals φ on A with $\varphi(1) = 1$, they comprise the *state space* $S(A)$, and the *numerical range* [12] is

$$W(x) = \{\varphi(x) : \varphi \in S(A)\}, \quad x \in A.$$

This is a convex and compact set of scalars. Some authors use not necessarily closed versions of the numerical range, such as $\{\langle x\zeta, \zeta \rangle : \zeta \in \text{Ball}(H)\}$ in the case x is an operator on Hilbert space H , but since these are dense in our $W(x)$ we avoid these. Let \mathbb{H} denote the open right half plane, with $\overline{\mathbb{H}}$ the closed right half plane. We write \mathfrak{r}_A for the *accretive* (or ‘real-positive’) elements in a unital Banach algebra A , i.e. those elements x with numerical range $W(x)$ in $\overline{\mathbb{H}}$. We say that x is *strictly accretive* if its numerical range is in \mathbb{H} . In a possibly nonunital operator algebra A on a Hilbert space H there is a unique unitization by Meyer’s theorem (see [7, Section 2.1]), which we can take to be $A + \mathbb{C}I_H$. Here we can define $\mathfrak{r}_A = A \cap \mathfrak{r}_{A^+}$, and we have $\mathfrak{r}_A = \{x \in A : x + x^* \geq 0\}$. Also, for invertible a , the spectrum of a is in the right half plane if and only if the spectrum of a^{-1} is in the right half plane. We write $\text{Ball}(X)$ for the set $\{x \in X : \|x\| \leq 1\}$, and set

$$\mathfrak{F}_A = \{x \in A : \|1 - x\| \leq 1\} = 1 + \text{Ball}(A)$$

for a unital Banach algebra A . There is an associated cone

$$\mathfrak{c}_A = \mathbb{R}^+ \mathfrak{F}_A,$$

and we have (see [8])

$$\mathfrak{r}_A = \overline{\mathbb{R}^+ \mathfrak{F}_A}.$$

By a *root* we mean a fractional power x^r where $r = \frac{1}{n}$ for $n \in \mathbb{N}$. See [6, Section 6] for a review of these. An element x of a unital Banach algebra A whose spectrum contains no real negative numbers nor 0, has a unique principal n th root in $\{x\}''$ for all $n \in \mathbb{N}$; that is a unique n th root with spectrum in the interior of the sector $S_{\frac{\pi}{n}}$; hence a unique square root with spectrum in the open right half plane \mathbb{H} (see [30, p. 360] for the square root case, which can be easily adapted for the n th root). We note that if x is an element of A whose numerical range $W(x)$ satisfies $W(x) \subset S_{\theta}$ for some $\theta < \pi$ then the formula of Stampfli and Williams [31, Lemma 1] and some basic trigonometry shows that x is sectorial of angle $\theta < \pi$ in the sense of e.g. [19], so that all the facts about roots of sectorial operators from that text apply.

Note that if a is invertible then $\text{Sp}(a^{-1}) = \{\lambda^{-1} : \lambda \in \text{Sp}(a)\}$, so that we have $(a^{-1})^{\frac{1}{2}} = (a^{\frac{1}{2}})^{-1}$ if $\text{Sp}(a)$ contains no real negative numbers. This follows from the unicity of principal roots mentioned above, because both have spectrum in a sector of angle $< \frac{\pi}{2}$.

It is well known that the accretive elements are closed under roots, or r th powers for $r \in (0, 1)$. Note too that $a \in \mathfrak{c}_A$ implies that $a^r \in \mathfrak{c}_A$ for such r . This is because \mathfrak{F}_A is closed under such powers (see e.g. [6, Proposition 6.3]). Also, in an operator algebra if $W(a) \subset S_\theta$ for $\theta < \pi$ then $W(a^r) \subset S_{r\theta}$ for $0 \leq r \leq 1$ (see e.g. [2, Corollary 4.6] for a more general result).

We will use Crouzeix's analytic functional calculus (e.g. [14, Theorem 2.1]). Crouzeix showed for example that there is a universal constant K (we will call this Crouzeix's constant), which is known to be smaller than 12, and is possibly 2, such that $\|f(x)\| \leq K \sup_{z \in W(x)} |f(z)|$, for any bounded operator x on a Hilbert space and rational function f with no poles in $W(x)$.

2. MORE BACKGROUND RESULTS

The following is no doubt well known (the formula is in Corollary 3.1.14 or 3.2.1 (d) in [19] in the case x is sectorial), and its proof follows a standard route. For example, it is similar to but easier than the case considered in [26], but since we do not know of an explicit reference we sketch the argument.

Lemma 2.1. *If x is an invertible element in a unital Banach algebra whose spectrum contains no real strictly negative numbers, and if $0 < \alpha < 1$, then*

$$x^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{-\alpha} (t1 + x)^{-1} dt.$$

In particular, $x^{-\frac{1}{2}} = \frac{2}{\pi} \int_0^\infty (t^2 1 + x)^{-1} dt$.

Proof. Let $R = \|x\|$, and choose θ less than but very close to π , and choose $\epsilon \geq 0$ small enough so that $\|(x - zI)^{-1} - x^{-1}\| < 1$ for $|z| < \epsilon$. Choose $r > R$. Consider the simple closed curve $\Gamma^{r, \epsilon, \theta}$, oriented counterclockwise, consisting of the numbers with argument in $[-\theta, \theta]$ which are on the two circles centered at 0 and having radii r and ϵ , together with the lines $z = \pm te^{i\theta}$ for $\epsilon \leq t \leq r$. By the Riesz functional calculus

$$x^{-\alpha} = \frac{1}{2\pi i} \int_\Gamma (z1 - x)^{-1} z^{-\alpha} dz.$$

The part of the integral over the small circular arc contributes something which in norm is less than $(\|x^{-1}\| + 1) \cdot \epsilon \cdot \epsilon^{-\alpha}$ to the integral. But this converges to 0 as $\epsilon \rightarrow 0$, and so letting $\epsilon \rightarrow 0$ we may replace $\Gamma^{r, \epsilon, \theta}$ by $\Gamma^{r, 0, \theta}$. Looking at the bottom half of $\Gamma^{r, 0, \theta}$, we may let $\theta \rightarrow \pi^-$, and hence the line segment part of the curve may be taken to lie on the negative x axis. However there is an issue with what becomes of $z^{-\alpha}$ as z approaches the negative real axis from below: if $z = te^{i\theta}$, for a number θ slightly larger than $-\pi$, then $z^{-\alpha} = t^{-\alpha} e^{-i\alpha\theta} \rightarrow t^{-\alpha} (\cos(\alpha\pi) + i \sin(\alpha\pi))$. Note that this is different to what happens with z on the 'upper line segment', here we will get a limit $t^{-\alpha} (\cos(\alpha\pi) - i \sin(\alpha\pi))$. The integral over the 'lower line segment' thus leads to a contribution of is

$$\frac{-1}{2\pi i} \int_0^r (-t - x)^{-1} (\cos(\alpha\pi) + i \sin(\alpha\pi)) t^{-\alpha} dt.$$

Similarly, the contribution from the 'upper' line segment can be seen to be

$$\frac{1}{2\pi i} \int_0^r (-t - x)^{-1} (\cos(\alpha\pi) - i \sin(\alpha\pi)) t^{-\alpha} dt,$$

and so the two line segments together contribute $\frac{\sin(\alpha\pi)}{\pi} \int_0^r (t + x)^{-1} t^{-\alpha} dt$. The circular part of Γ is distance greater than $r - R$ from the numerical range of x and

so by [31, Lemma 1] it contributes at most $\frac{r^{1-\alpha}}{r-R}$. But this converges to 0 as $r \rightarrow \infty$. Thus letting $r \rightarrow \infty$ we obtain the desired formula. If $\alpha = \frac{1}{2}$ we can let $u = \sqrt{t}$ to obtain the second formula. \square

Most of the following is also well known to experts (see e.g. [13, 18]).

Lemma 2.2. *If A is a unital operator algebra on a Hilbert space and if $x \in \mathfrak{r}_A$ is an invertible accretive operator in A then $x^{-1} \in \mathfrak{r}_A$. That is, inverses of invertible accretive operators on a Hilbert space are accretive. More generally, if $W(x) \subset S_\theta$ then $W(x^{-1}) \subset S_\theta$ if $0 \leq \theta \leq \frac{\pi}{2}$ and x is invertible. Finally, if a is an invertible in A which is strictly accretive (this is equivalent for invertibles to being in \mathfrak{c}_A), then a^{-1} is strictly accretive (or equivalently, in \mathfrak{c}_A).*

Proof. Throughout this proof let x be invertible and accretive. For the first statement (which is well known but since it is short we will prove), suppose that $A \subset B(H)$ is a unital subalgebra. Then any $\eta \in H$ equals $x\zeta$ with $\zeta \in H$ and

$$\operatorname{Re} \langle x^{-1}\eta, \eta \rangle = \operatorname{Re} \langle \zeta, x\zeta \rangle = \operatorname{Re} \langle x\zeta, \zeta \rangle \geq 0.$$

So x^{-1} is accretive. The second statement is in the references cited above the lemma.

Now $a \in \mathfrak{c}_A$ iff there exists $t > 0$ with $\|1 - ta\|^2 \leq 1$, which is easy to see via the C^* -identity happens iff $a + a^* \geq ta^*a$. This in turn is equivalent to $a + a^* \geq \epsilon 1$ for some $\epsilon > 0$, since a is invertible. Then [13, Proposition 3.5] implies that $a^{-1} \in \frac{1}{\epsilon} \mathfrak{F}_A \subset \mathfrak{c}_A$. \square

The last lemma is not true for unital Banach algebras. For example in ℓ_2^1 with the usual convolution product, $(1+i, 1)$ is accretive, but its inverse $\frac{1}{-1+2i}(1+i, -1)$ is not accretive, using the criterion for being accretive given in Example 3.14 in [8].

Remark 2.3. The last observation gives one way to see that the Cayley transform $\kappa(x)$ and the transform $\mathfrak{F}(x)$ considered e.g. in [11, Section 2.2], are not contractions for accretive x in general unital Banach algebras. Indeed if $\kappa(x)$ was contractive then $\mathfrak{F}(x) = \frac{1}{2}(1 + \kappa(x))$ is contractive, and hence

$$\|(t + x^{-1})^{-1}\| = \frac{1}{t} \mathfrak{F}(tx) \leq \frac{1}{t}, \quad t > 0.$$

This implies that $x^{-1} \in \mathfrak{r}_A$ by e.g. [6, Lemma 2.4].

We will say that an element x in a unital Banach algebra A is *type M* if there exists a constant M such that $\|(t1 + x)^{-1}\| \leq M/t$ for all $t > 0$. This is essentially what is called being sectorial in [19] (see p. 20–21 there, replacing a by left multiplication by a in $B(A)$), or sometimes called being ‘non-negative’. We use this older name simply because there is ambiguity in the literature: e.g. we (and many others) have used the word sectorial for the stronger notion of an operator whose numerical range is contained in a sector S_θ with $\theta < \pi$. Note that the inequality $\|(t1 + x)^{-1}\| \leq 1/t$ for all $t > 0$ is equivalent to x being accretive (see e.g. [6, Lemma 2.4]). Inverses of invertible type M elements are type M by an elementary equality for inverses [19, Proposition 2.1.1]. It is well known that if the spectrum of an invertible element a contains no real strictly negative numbers then a is type M. Indeed for any $a \in A$ the identity defining ‘type M elements’ is always true for $t > 2\|a\|$ by an inequality in the proof of the Neumann series lemma:

$$\|(1 + a/t)^{-1}\| \leq 1/(1 - \|a/t\|) \leq 2;$$

and $t\|(t1+a)^{-1}\|$ is continuous and hence bounded on $[0, 2\|a\|]$. (We remark though that this is false if a is not invertible. Consider for example $a(z) = z$ on the parabola $y = x^2$ for $|x| \leq 1$, here $\|t(t1+a)^{-1}\|$ dominates $1/t$, the absolute value of $t(t1+a)^{-1}$ at $z = -t + it^2$. Also, there are type M elements a with negative numbers in $W(a)$, e.g. most invertible 2×2 matrices with -1 in the 1-1 entry.)

Lemma 2.4. *In a Banach algebra, if a, b are type M then $\|a^t - b^t\| \leq K\|a - b\|^t$ for all $t \in (0, 1]$, for a constant K depending on t .*

Proof. This follows from the proof of the analogous result in [27]. \square

Some details seem to be skipped in the proof of uniqueness in [26, Theorem 2.8], which with the help of [29] we supply below, also slightly improving the result. See also [28, Chapter 6].

Theorem 2.5. *If A is a unital Banach algebra, $m \in \mathbb{N}$, and $x \in A$ is such that $W(x)$ contains no strictly negative numbers, then x has a unique m th root with spectrum in $S_{\frac{\pi}{m}}$. This root is in the closed subalgebra generated by x .*

Also we have $(e^{i\theta}x)^s = e^{is\theta}x^s$ for $s \in [0, 1]$ and $|\theta| \leq \pi$, provided that $W(e^{i\rho}x)$ contains no strictly negative numbers for all ρ between 0 and θ (including 0 and θ).

Proof. If $W(x) \subset S_\beta$ for some $\beta < \pi$, then x is type M as stated above, and then the first part of the result (except for the ‘subalgebra generated’ assertion) is in [29] (the main part being in [26] too). We will take this for granted in the following argument. In the contrary case, since $W(x)$ is convex, it follows that $W(x) \subset i\mathbb{H}$ or $W(x) \subset -i\mathbb{H}$. We assume the first, the second being similar. Then $i^{\frac{1}{m}}(-ix)^{\frac{1}{m}}$ is an m th root of x with spectrum in $i^{\frac{1}{m}}S_{\frac{\pi}{2m}} \subset S_{\frac{\pi}{m}}$. That $x^{\frac{1}{m}}$ is in the closed subalgebra generated by x may be found e.g. in the discussion after Proposition 6.3 in [6].

Now suppose that c_1, c_2 are two m th roots of x with spectrum in $S_{\frac{\pi}{m}}$. Then for $\epsilon > 0$ let $d_k = c_k + \epsilon 1$, then d_k^m is invertible and has spectrum containing no strictly negative numbers by the spectral mapping theorem. Thus d_k^m is type M by an observation above Lemma 2.4, and so we can use the argument in [26, 29]: by an argument in [27] (see Lemma 2.4 above) we have

$$\|c_1 - c_2\| \leq K\|d_1^m - d_2^m\| \rightarrow 0$$

as $\epsilon \rightarrow 0$, so $c_1 = c_2$.

For the last assertion, let θ be as described. By writing $\theta = p \frac{\theta}{p}$ for a large integer p and iterating the identity we are proving p times, we may assume that θ is as close to 0 as we like. In fact, the case that $-\frac{\pi}{2} \leq \theta < 0$ and $e^{i\theta}x$ is accretive is done in [2, Corollary 4.6] (note that the first centered equation on page 564 there also follows from the uniqueness argument just after the next centered equation there). Next suppose that the largest argument of numbers in $W(x)$ is $\alpha > \frac{\pi}{2}$, and suppose that $\frac{\pi}{2} - \alpha < \theta < 0$, so that $W(e^{i\theta}x)$ still intersects the interior of the third quadrant. Choose $\rho > 0$ such that $W(e^{i(\theta-\rho)}x)$ is accretive, then by the case just discussed we have $(e^{i(\theta-\rho)}x)^s = e^{is(\theta-\rho)}x^s$, so that

$$e^{is\theta}x^s = e^{is\rho}(e^{i(\theta-\rho)}x)^s = (e^{i\rho}e^{i(\theta-\rho)}x)^s = (e^{i\theta}x)^s,$$

where in the second last equality we used the case from [2] again. The next case we consider is if x is accretive, and $\theta < 0$. Let $a = e^{i\theta}x$, then $e^{-i\theta}a = x$. By the case from [2] we have $(e^{-i\theta}a)^s = e^{-is\theta}a^s$, so that $e^{is\theta}x^s = (e^{i\theta}x)^s$ as desired. Next, if $W(x)$ contains numbers in the interior of the third quadrant and θ negative but

very small, choose $\rho > 0$ with $e^{i(\theta+\rho)}x$ accretive. By the case from [2], we have $(e^{i(\theta+\rho)}x)^s = e^{is(\theta+\rho)}x^s$, so that

$$e^{is\theta}x^s = e^{-is\rho}(e^{i(\theta+\rho)}x)^s = (e^{-i\rho}e^{i(\theta+\rho)}x)^s = (e^{i\theta}x)^s,$$

similarly to a case above. Finally, if $\theta > 0$, replace x by $a = e^{i\theta}x$ and θ with its negative, and apply the above. \square

The last assertion of the last result is also no doubt known to some experts, see e.g. [24, Theorem 10.1]. The following is no doubt also known, but again we do not know of a reference for it as stated.

Corollary 2.6. *If a is a Hilbert space operator with no strictly negative numbers in $W(a)$, and with the arguments of numbers in $W(a)$ inside $[\alpha, \beta]$ for $-\pi \leq \alpha \leq \beta \leq \pi$, then for $s \in (0, 1)$ the arguments of numbers in $W(a^s)$ are in $[s\alpha, s\beta]$.*

Proof. Let $\nu = \frac{\beta-\alpha}{2}, \theta = \frac{\beta+\alpha}{2}$, then $W(e^{-i\theta}a) \subset S_\nu$. Hence using the last assertion of the last result, $W(e^{-is\theta}a^s) = W((e^{-i\theta}a)^s) \subset S_{s\nu}$, so that the arguments of numbers in $W(a^s)$ are in $[-s\nu + s\theta, s\nu + s\theta] = [s\alpha, s\beta]$. \square

In [6, Section 6] we gave an estimate for the ‘sectorial angle’ of $W(x^t)$ for accretive elements in a Banach algebra. The following is the variant of that result in the case that $W(x) \subset S_\theta$ for $\frac{\pi}{2} < \theta < \pi$.

Lemma 2.7. *If A is a unital Banach algebra and if $x \in A$ has no negative numbers in its numerical range and satisfies $W(x) \subset S_{\frac{\pi}{2}+\theta}$, where $0 \leq \theta \leq \frac{\pi}{2}$, then $W(x^{\frac{1}{p}}) \subset S_{\frac{\pi}{2}+\frac{\theta}{p}}$ for $p \in \mathbb{N}$. If A is also an operator algebra on a Hilbert space then $W(x^{\frac{1}{p}}) \subset S_{\frac{\pi}{2p}+\frac{\theta}{p}}$.*

Proof. We have that $e^{-i\theta}x$ is accretive, so that $e^{-i\frac{\theta}{p}}x^{\frac{1}{p}}$ is accretive (see also the proof of Theorem 2.5). Hence $W(x^{\frac{1}{p}}) \subset S_{\frac{\pi}{2}+\frac{\theta}{p}}$. The Hilbert space case is well known (see e.g. [26, Theorem 2.8] and the last section in [2]). \square

Proposition 2.8. *Let $t > 1$. In a unital Banach algebra A if $\|1 - tx\| \leq 1$ and $\|1 - x\| = 1$ then every functional that achieves its norm at $1 - x$ is a scalar multiple of a state. Hence if x is an element of A with $0 \notin W(x)$ and with $\|1 - tx\| \leq 1$ for some $t > 1$ then $\|1 - x\| < 1$.*

Proof. Any norm 1 functional f with $f(1 - x) = 1$, satisfies

$$1 \geq |f(1 - tx)| = |f(1 - t + t(1 - x))| = |t - (t - 1)f(1)| \geq t - (t - 1)|f(1)| \geq t - (t - 1) = 1.$$

So these are all equalities. It is clear that $f(1) \neq 0$. By the converse to the triangle inequality, the second last and the last (in)equality implies that $f(1) \geq 0$ and then that $f(1) = 1$. So f is a state.

For the second assertion, by convexity $\|1 - x\| \leq 1$. if $\|1 - x\| = 1$ then by the first assertion there is a state that achieves its norm at $1 - x$, so $f(x) = 0$ contradicting $0 \notin W(x)$. \square

3. THE ‘SIGN’ OF A BANACH ALGEBRA ELEMENT

In this section we point out that much of the theory of the ‘sign of a matrix’ summarized in [20, Chapter 5] (this is sometimes called the ‘sector’) generalizes to Banach algebras or operator algebras. We will follow the development in [20, Chapter 5] slavishly—our intent is simply to repeat the results that generalize, and in each case say a word about how the proof needs to be adapted if necessary.

By the spectral mapping theorem, if x is an element of a unital Banach algebra with $\text{Sp}(x) \cap i\mathbb{R} = \emptyset$, then $\text{Sp}(x^2)$ contains no real negative numbers nor 0. So as we said in the Introduction, x^2 has a unique principal square root, whose inverse we write as $(x^2)^{-\frac{1}{2}}$. We define

$$\text{sign}(x) = x(x^2)^{-\frac{1}{2}} \quad \text{if } \text{Sp}(x) \cap i\mathbb{R} = \emptyset.$$

As in the matrix theory, $\text{sign}(x)$ has an integral formula

$$\text{sign}(x) = \frac{2x}{\pi} \int_0^\infty (t^2 1 + x^2)^{-1} dt.$$

This follows immediately from Lemma 2.1.

Proposition 3.1. *Suppose that a is an element of a unital Banach algebra A with $\text{Sp}(a) \cap i\mathbb{R} = \emptyset$, and let $S = \text{sign}(a)$.*

- (1) $S^2 = 1$.
- (2) $S \in \{a\}''$.
- (3) *If a is also a selfadjoint Hilbert space operator then S is a symmetry (that is, a selfadjoint unitary). More generally, $\text{sign}(a^*) = \text{sign}(a)^*$.*
- (4) $E_+ = \frac{1}{2}(I + S)$ and $E_- = \frac{1}{2}(I - S)$ are idempotents with sum 1, and with $SE_+ = E_+$, $SE_- = -E_-$, and $S = E_+ - E_-$. Indeed E_+ is the spectral idempotent [15] of a associated with $\text{Sp}(a) \cap \mathbb{H}$.
- (5) $\text{Sp}(a) \subset \mathbb{H}$ iff $1 = \text{sign}(a)$.
- (6) $\text{sign}(v^{-1}av) = v^{-1}\text{sign}(a)v$ if v is an invertible element of the algebra.
- (7) $a = \text{sign}(a)N$ where $N = (a^2)^{\frac{1}{2}}$.
- (8) $\text{sign}(ca) = \text{sign}(c)\text{sign}(a)$ if c is a nonzero real scalar.
- (9) $\text{sign}(a^{-1}) = \text{sign}(a)$.

Proof. (1) and (7) are obvious, and (2) is clear since the square root is in $\{a\}''$. For (3) use the fact that $*$ ‘commutes’ with the inverse, and with the square root (we leave the latter as a simple exercise using the uniqueness of the primary square root). The first assertions in (4) follow from (1). The ‘spectral idempotent’ assertion is because working with respect to the Banach algebra generated by 1 and a , if χ is a character of A with $\chi(a) \in \mathbb{H}$ then $\chi(a) \cdot (\chi(a)^2)^{-\frac{1}{2}} = 1$. And if χ is a character with $\chi(a) \in -\mathbb{H}$ then $\chi(a) \cdot (\chi(a)^2)^{-\frac{1}{2}} = -1$.

Since $\text{Sp}(a) \subset \mathbb{H}$ iff $(a^2)^{\frac{1}{2}} = a$, item (5) is clear. For (6),

$$\text{sign}(v^{-1}av) = (v^{-1}av)(v^{-1}a^2v)^{-\frac{1}{2}} = v^{-1}\text{sign}(a)v.$$

We are silently using the uniqueness property of the principal square root here. We leave (8) as an exercise, and (9) is simple algebra using the relations $a \cdot a = (a^2)^{\frac{1}{2}} \cdot (a^2)^{\frac{1}{2}}$ and $((a^2)^{\frac{1}{2}})^{-1} = ((a^2)^{-1})^{\frac{1}{2}} = ((a^{-1})^2)^{\frac{1}{2}}$. One may also deduce (9) from Theorem 3.4 below. \square

Proposition 3.2. *For operators a, b on a Hilbert space such that $\text{Sp}(ba)$ contains no negative real numbers nor zero, we have*

$$\text{sign} \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c^{-1} & 0 \end{bmatrix}$$

where $c = a(ba)^{-\frac{1}{2}}$.

Proof. Since it is well known that $\text{Sp}(ab) \setminus \{0\} = \text{Sp}(ba) \setminus \{0\}$, we also have that $\text{Sp}(ab)$ contains no negative real numbers nor zero. Using graduate level operator theory it is clear that the rest of the proof of [20, Theorem 5.2] works in infinite dimensions. \square

Remark 3.3. 1) It is clear that Proposition 3.2 works for Banach algebras too for any appropriate norm on $M_2(A)$.

2) It is no doubt true as in the matrix case that $\text{sign}(a) = \frac{2}{\pi} \lim_{t \rightarrow \infty} \arctan(ta)$ for any element a of a unital Banach algebra A with $\text{Sp}(a) \cap i\mathbb{R} = \emptyset$. Indeed this boils down to showing that $\int_0^t (s^2 1 + a^2)^{-1} ds = \arctan(ta)$ for positive scalars t , and the latter is possibly well known.

It follows that for an invertible operator a on a Hilbert space with no negative numbers in its spectrum, we have

$$\text{sign} \begin{bmatrix} 0 & a \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & a^{\frac{1}{2}} \\ a^{-\frac{1}{2}} & 0 \end{bmatrix}.$$

We now turn to the (iterative) Newton method $X_{k+1} = \frac{1}{2}(X_k + X_k^{-1})$ for $\text{sign}(a)$. We will take $X_0 = a$.

Theorem 3.4. *Suppose that a is an element of a unital Banach algebra with $\text{Sp}(a) \cap i\mathbb{R} = \emptyset$, and let $S = \text{sign}(a)$. Then the Newton iterates X_k above for $\text{sign}(a)$ converge quadratically to S , and also $X_k^{-1} \rightarrow S$, with*

$$\|X_{k+1} - S\| \leq \frac{1}{2} \|X_k^{-1}\| \|X_k - S\|^2,$$

and $X_k = (1 - G_0^{2^k})^{-1}(1 + G_0^{2^k})S$ for $k \geq 1$, where $G_0 = \kappa(N)$, where $N = (a^2)^{\frac{1}{2}}$.

Proof. We adjust the proof in [20, Theorem 5.2] slightly, and omit several easy details. By the spectral mapping theorem, since the spectrum of N lies in the open right half plane, the spectrum of G_0 lies in the open unit ball, and hence also the spectrum of $G_0^{2^k}$ lies in this ball. So $(1 - G_0^{2^k})^{-1}$ exists. Set $X_k = (1 - G_0^{2^k})^{-1}(1 + G_0^{2^k})S$; we will show that $X_{k+1} = \frac{1}{2}(X_k + X_k^{-1})$. Indeed $\frac{1}{2}(X_k + X_k^{-1}) = \frac{S}{2}((1 - G_0^{2^k})^{-1}(1 + G_0^{2^k}) + (1 - G_0^{2^k})(1 + G_0^{2^k})^{-1})$ equals

$$\frac{S}{2}(1 - G_0^{2^k})^{-1}(1 + G_0^{2^k})^{-1}[(1 - G_0^{2^k})^2 + (1 + G_0^{2^k})^2] = \frac{S}{2}(1 - G_0^{2^{k+1}})^{-1}[2(1 + G_0^{2^{k+1}})]$$

which equals X_{k+1} . Since the spectral radius of G_0 is smaller than 1, it follows that $G_0^{2^k} \rightarrow 0$ as $k \rightarrow \infty$, so that $X_k = (1 - G_0^{2^k})^{-1}(1 + G_0^{2^k})S \rightarrow S$ (we are using the continuity of the inverse at 1 in a Banach algebra). Similarly, $X_k^{-1} = (1 - G_0^{2^k})(1 + G_0^{2^k})^{-1}S \rightarrow S$. The rest is as in [20, Theorem 5.6]. \square

Remark 3.5. A common application of the sign function for matrices in numerical analysis and engineering is to solve $ax - xb = y$ for x . Suppose that the spectrum of a is in the negative right half plane and the spectrum of b is in the positive right half plane. As on [4, p. 11], we have

$$\begin{bmatrix} a & y \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

The sign of the matrix in the middle is the diagonal matrix with diagonal entries 1 and -1 , and so it follows from Proposition 3.1 (6) that

$$\text{sign}\left(\begin{bmatrix} a & y \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2x \\ 0 & -1 \end{bmatrix}.$$

Thus x is one half of the 1-2 entry of $\text{sign}\left(\begin{bmatrix} a & y \\ 0 & b \end{bmatrix}\right)$.

4. NEWTON'S METHOD FOR THE SQUARE ROOT

Newton's method for the square root $a^{\frac{1}{2}}$ is

$$X_{k+1} = \frac{1}{2}(X_k + X_k^{-1}a),$$

with $X_0 = I$ usually.

Define $\kappa(\lambda) = \frac{\lambda-1}{\lambda+1}$ for $\lambda \in \mathbb{C}, \lambda \neq -1$. This map takes the right half plane onto the unit circle (omitting the number 1). The inverse of this is the map $\kappa^{-1}(\lambda) = \frac{1+\lambda}{1-\lambda}$. (Some authors use instead the map $\lambda \mapsto \frac{1-\lambda}{1+\lambda}$, which is its own inverse.)

Lemma 4.1. *Fix $n \in \mathbb{N}$. The supremum of $\frac{t\kappa(t)^{2^n}}{1-\kappa(t)^{2^n}}$ on $(0, 1]$ is $\frac{1}{2^{n+1}}$, which it converges to as $t \rightarrow 0$.*

Proof. To see this, let us change variables, letting $s = -\kappa(t)$, so that $t = -\kappa(s)$. Then the function to be maximized is $\frac{|\kappa(s)|s^k}{1-s^k}$, for $s \in [0, 1)$ and $k = 2^n$. We claim that this is an increasing function. Indeed if one takes its derivative, the denominator is positive as usual, and the numerator on $(0, 1)$ is a positive multiple of $(-2s + k(1-s^2))(1-s^k) + k(1-s^2)t^k$, and the latter equals

$$k(1-s^2) - 2s(1-s^k) \geq (1-s)[k(1+s) - 2ks] = k(1-s)^2 \geq 0,$$

since $2ks(1-s) \leq 2s(1-s^k)$. Thus the function is increasing, and its supremum is its limit as $t \rightarrow 1^-$, which by L'Hopitals rule is $\frac{1}{2^{n+1}}$. \square

We now turn to the square root, which has many equivalent definitions (see e.g. [6, Section 6]). For example it has formula

$$x^{\frac{1}{2}} = \frac{2}{\pi} x \int_0^\infty (t^2 1 + x)^{-1} dt,$$

if x is type M (by substituting $u = t^{\frac{1}{2}}$ in the Balakrishnan formula (3.2) in [19]), or if x is invertible and the spectrum of x contains no real strictly negative numbers (by Lemma 2.1).

Theorem 4.2. *Suppose that a is an element of a unital Banach algebra A with $\text{Sp}(a)$ containing no negative real numbers nor 0. Suppose that $X_0 \in \{a\}'$ with*

$\text{Sp}(a^{-\frac{1}{2}}X_0)$ contained in the open right half plane. Then the Newton iterates X_k above for the square root converge quadratically to $a^{\frac{1}{2}}$, and also $X_k^{-1} \rightarrow a^{-\frac{1}{2}}$, with

$$\|X_{k+1} - a^{\frac{1}{2}}\| \leq \frac{1}{2}\|X_k^{-1}\|\|X_k - a^{\frac{1}{2}}\|^2,$$

and $X_k = a^{\frac{1}{2}}(1 - G_0^{2^k})^{-1}(1 + G_0^{2^k})S$ for $k \geq 1$, where $G_0 = \kappa(N)$, where $N = ((a^{-\frac{1}{2}}X_0)^2)^{\frac{1}{2}}$.

Proof. The proof in [20, Theorem 6.9] works in our setting too, using our Theorem 3.4 in place of [20, Theorem 5.2]. \square

Remark 4.3. We point out that if A is an operator algebra then in the situation of Theorem 4.2 we also get that if X_0 and $X_0^{-2}a$ are accretive, then X_k and $X_k^{-2}a$ are accretive, and $X_k^{-1}a^{\frac{1}{2}}$ has numerical range in $S_{\frac{\pi}{4}}$, for all k . We prove this by induction. If it is true for k then

$$X_{k+1}^2 a^{-1} = \frac{1}{4}(X_k^2 a^{-1} + 2 \cdot 1 + X_k^{-2} a).$$

All three parts of this are accretive, using Lemma 2.2. So $X_{k+1}^2 a^{-1}$ is accretive, and so also is $X_{k+1}^{-2} a$ by Lemma 2.2. Also, $X_{k+1}^{-1} a^{\frac{1}{2}}$ has spectrum in the right half plane as we shall see soon (around Equation (4.3) below), so $X_{k+1}^{-1} a^{\frac{1}{2}}$ is the principal square root of $X_{k+1}^2 a^{-1}$ and has numerical range in $S_{\frac{\pi}{4}}$. Then $X_{k+1} = \frac{1}{2}(X_k + (X_k^{-1} a^{\frac{1}{2}}) a^{\frac{1}{2}})$. Now the product of two commuting operators with numerical range in $S_{\frac{\pi}{4}}$ is accretive [2]. Hence X_{k+1} is accretive, being the average of two accretives.

We next discuss Newton's method for noninvertible a . This works for a rather large class of elements in operator algebras. We will usually take $X_0 = 1$ or $X_0 = (a + 1)/2$ (note that if $X_0 = 1$ then $X_1 = (a + 1)/2$, so we may as well assume $X_0 = 1$). An important remark about other 'starting values' is stated just after Proposition 7.3.

Theorem 4.4. *If a is an operator on a Hilbert space with numerical range $W(a) \subset S_\theta$ for some $\theta < \pi$, then Newton's method for the square root, with $X_0 = 1$ or $X_0 = (a + 1)/2$, converges to the principal square root $a^{\frac{1}{2}}$. Indeed for n large enough, the n th iterate X_n in Newton's method has distance less than $\frac{C_\rho K}{2^n}$ from $a^{\frac{1}{2}}$. Here K is Crouziera's constant (mentioned at the end of the introduction), and C_ρ is any constant greater than $\sec(\frac{\rho}{2})$ where ρ is the sectorial angle of a (thus $W(a) \subset S_\rho$). In particular, if a is accretive then $\|X_n - a^{\frac{1}{2}}\| \leq \frac{K}{2^{n-1}}$ for all n large enough.*

Proof. First we work in any unital Banach algebra. Let $c = a^{\frac{1}{2}}$, whose spectrum is contained in a sector S_θ where $\theta < \frac{\pi}{2}$ (see Theorem 2.5). For now let X_0 be any invertible in the algebra with the property that $d = X_0^{-1}c$ satisfies that $\text{Sp}(d) \setminus \{0\}$ is in the open right half plane (this is clearly true if $X_0 = 1$ (and we will see that it is true if $X_0 = (a + 1)/2$ and hence also if $X_0 = a + 1$)). Let $G_0 = (1 - d)(1 + d)^{-1}$. This is the negative of the Cayley transform $\kappa(d)$ of d . We note that 1 is in the spectrum of G_0 if c is not invertible. However -1 is never in the spectrum of G_0 . Indeed the following is true:

Claim: 1 is the only number in the spectrum of G_0 which has modulus 1. The elements in the spectrum of G_0 with modulus 1 correspond, by the spectral mapping

theorem, to elements in the spectrum of $\kappa(d)$ with modulus 1, and these correspond to purely imaginary elements (or 0) in the spectrum of d . By our hypothesis on d above only 0 is possible. However the latter 0 would correspond to 1 in the spectrum of G_0 , not to -1 .

From the Claim it follows also that $G_k = G_0^{2^k}$ does not have -1 in its spectrum.

We next claim that X_n is invertible and in fact

$$(4.1) \quad X_n = \frac{(X_0 + c)}{2}(1 + G_0^{2^n})[(1 + G_0)(1 + G_0^2) \cdots (1 + G_0^{2^{n-1}})]^{-1}, \quad n \in \mathbb{N}.$$

We prove (4.1) by induction. We leave it to the reader to check the case $n = 1$. Assume it is true for n . We use the polynomial identity

$$(4.2) \quad (1 - z) \prod_{k=0}^{n-1} (1 + z^{2^k}) = 1 - z^{2^n}$$

setting $z = G_0$. Note that $1 - G_0 = 2c(X_0 + c)^{-1}$, so that $2c(X_0 + c)^{-1}[(1 + G_0)(1 + G_0^2) \cdots (1 + G_0^{2^{n-1}})] = 1 - G_0^{2^n}$. Now $X_n^{-1}c$ equals

$$2c(X_0 + c)^{-1}(1 + G_0^{2^n})^{-1}[(1 + G_0)(1 + G_0^2) \cdots (1 + G_0^{2^{n-1}})] = (1 + G_0^{2^n})^{-1}(1 - G_0^{2^n}).$$

That is,

$$(4.3) \quad X_n^{-1}c = -\kappa(G_0^{2^n}).$$

By the spectral mapping theorem and what we said earlier about elements in the spectrum of G_0 with modulus 1, it follows that $\text{Sp}(X_n^{-1}c) \setminus \{0\}$ is contained in the open right half plane. We remark in passing that in the Hilbert space operator case and $X_0^{-1}c$ is accretive (which is true if e.g. $X_0 = 1$), then by the theory of the Cayley transform G_0 is a contraction, hence $\| -G_0^{2^n} \| \leq 1$, and so $X_n^{-1}c$ is accretive. As we saw earlier, if a is an invertible operator on a Hilbert space and $X_0 = 1$ then $W(X_n^{-1}c) \subset S_{\frac{\pi}{4}}$ for all n . (We imagine that this should be true even if a is not invertible.)

Thus

$$X_{n+1} = \frac{1}{2}(X_n + X_n^{-1}a) = \frac{X_n}{2}(1 + (X_n^{-1}c)^2) = \frac{X_n}{2}(1 + \kappa(G_0^{2^n})^2),$$

which equals $\frac{X_n}{2}(2(1 + G_0^{2^{n+1}})(1 + G_0^{2^n})^{-2})$, using the easily checked identity $1 + \kappa(w)^2 = 2(1 + w^2)(1 + w)^{-2}$, which is true for any w with $1 + w$ invertible. Thus

$$X_{n+1} = \frac{(X_0 + c)}{2}(1 + G_0^{2^{n+1}})[(1 + G_0)(1 + G_0^2) \cdots (1 + G_0^{2^n})]^{-1}$$

as desired in the induction step.

Suppose that $X_0 = p(c)$ where $p(z)$ is a nonvanishing analytic function on a neighborhood of the spectrum of c . Our assumption on d above follows if $q(z) = z/p(z)$ is in the open right half plane for all $z \in \text{Sp}(c) \setminus \{0\}$. This in turn follows for example if a is accretive (so that $W(c) \subset S_{\frac{\pi}{4}}$) and if $p(\text{Sp}(c)) \subset S_{\frac{\pi}{4}}$. We thus have $X_n - c = f_n(c)$ where

$$f_n(z) = \frac{(p(z) + z)}{2}(1 + (\kappa \circ q)^{2^n})[(1 + \kappa(q(z)))(1 + \kappa(q(z))^2) \cdots (1 + \kappa(q(z))^{2^{n-1}})]^{-1} - z.$$

This is a rational function. Indeed using the polynomial identity (4.2) we have

$$(4.4) \quad f_n(z) = \frac{z(1 + \kappa(q(z))^{2^n})}{1 - \kappa(q(z))^{2^n}} - z = \frac{2z \kappa(q(z))^{2^n}}{1 - \kappa(q(z))^{2^n}}, \quad \text{Re } z > 0,$$

and $f_n(0) = \frac{p(0)}{2^n}$. (We note that assuming that $q(z) = z/p(z)$ is in the open right half plane for all $z \in \text{Sp}(c) \setminus \{0\}$, forces $|\kappa(q(z))| = 1$ only when $q(z) = 0$, that is, only when $z = 0$. The question is whether $f_n(c) \rightarrow 0$ as $n \rightarrow \infty$. This would follow from the continuity of the functional calculus *if* all of the f_n were analytic on a fixed neighborhood of 0, but unfortunately that is not generally the case.)

We remark that if $X_0 = 1$ then $G_0 = (1 - c)(1 + c)^{-1}$, the negative of the Cayley transform $\kappa(c)$ of c . Equation (4.1) becomes

$$(4.5) \quad X_n = \frac{1+c}{2} (1+G_0^{2^n})(1+c)^2[(1+G_0)(1+G_0^2) \cdots (1+G_0^{2^{n-1}})]^{-1}, \quad n \in \mathbb{N}.$$

We still have $X_n - c = f_n(c)$, but the formula for f_n in this case (c.f. the centered formula a few lines above Equation (4.4)) becomes

$$(4.6) \quad f_n(z) = \frac{1+z}{2} (1+\kappa(z)^{2^n})[(1+\kappa(z))(1+\kappa(z)^2) \cdots (1+\kappa(z)^{2^{n-1}})]^{-1} - z.$$

and so again using the polynomial identity (4.2) Equation (4.4) becomes

$$(4.7) \quad f_n(z) = \frac{z(1+\kappa(z)^{2^n})}{1-\kappa(z)^{2^n}} - z = \frac{2z\kappa(z)^{2^n}}{1-\kappa(z)^{2^n}}, \quad \text{Re } z > 0,$$

and $f_n(0) = \frac{1}{2^n}$. Again, the question is whether $f_n(c) \rightarrow 0$ as $n \rightarrow \infty$, which would follow from the continuity of the Riesz functional calculus *if* all of the f_n were analytic on a fixed neighborhood of 0, but unfortunately that is not the case. However if we are in an operator algebra then one may use a variant of the functional calculus for spectral sets, for example Crouzeix's analytic functional calculus mentioned at the end of the introduction. This we now do.

Henceforth, assume we are in an operator algebra, and that $X_0 = 1$. The numerical range $W(c)$ is contained in S_θ where $\theta = \rho/2$ (see e.g. [26, 2]). We will assume for clarity that a is accretive, and so we may take $\theta = \frac{\pi}{4}$, the case that $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ will be discussed at the end. It is a well known result of Crouzeix that the numerical range of any operator is a *K-spectral set* for a positive constant $K < 12$. Thus $\|f_n(c)\| \leq K\|f_n\|_{W(c)}$ for a constant K depending on the shape of (a closed region containing) the numerical range of c . We will estimate $\|f_n\|_E$ where E is the sector of the circle of radius $\|c\|$ contained in S_θ , and hence see that $\|f_n\|_{W(c)} \leq \|f_n\|_E \rightarrow 0$ as $n \rightarrow \infty$. It is easy to see that f_n has limit $\frac{1}{2^{n+1}}$ as one approaches 0 from the right. Fix a small $\delta > 0$. If one considers the picture of the image of E under the map $z \mapsto \frac{1-z}{1+z}$, one sees that $|\kappa(z)| < 1 - \delta$ for all $z \in E \setminus D(0, \epsilon)$, for a small $\epsilon > 0$ (independent of n). Hence for such z we have

$$|f_n(z)| \leq \frac{2|z||\kappa(z)|^{2^n}}{1-|\kappa(z)|^{2^n}} \leq \frac{2\|c\|(1-\delta)^{2^n}}{1-(1-\delta)^{2^n}}$$

The right side will be less than $\frac{1}{2^n}$ for n large enough, and so we see that for n large enough, the maximum of $|f_n(z)|$ is achieved on $E \cap D(0, \epsilon)$. By a similar argument, and the maximum modulus theorem, the maximum of $|f_n(z)|$ is achieved on the boundary lines of $E \cap D(0, \epsilon)$, and by symmetry on the upper of these two lines. Thus if $\theta = \frac{\pi}{4}$ we may assume that $z = t(1+i)$ for $0 < t < \epsilon$. Using the identity

$$|\kappa(z)|^2 = \frac{1 - 2\text{Re } z + |z|^2}{1 + 2\text{Re } z + |z|^2} = -\kappa\left(\frac{2\text{Re } z}{1 + |z|^2}\right),$$

we see that

$$(4.8) \quad |f_n(z)| \leq \frac{2|z||\kappa(z)|^{2^n}}{1-|\kappa(z)|^{2^n}} = \frac{2\sqrt{2}t\kappa(s)^{2^{n-1}}}{1-\kappa(s)^{2^{n-1}}} = \frac{\sqrt{2}s\kappa(s)^{2^{n-1}}}{1-\kappa(s)^{2^{n-1}}}(1+2t^2),$$

where $s = \frac{2t}{1+2t^2}$. By Lemma 4.1 the supremum of the last function is $\leq \frac{\sqrt{2}}{2^n}(1+2t^2) < \frac{1}{2^{n-1}}$ if $2\epsilon^2 < \sqrt{2}-1$. Thus given $\epsilon > 0$ we see that for n large enough we have $\|f_n\|_E \leq \frac{\sqrt{2}}{2^n}(1+\epsilon) < \frac{1}{2^{n-1}}$. Hence

$$\|X_n - a^{\frac{1}{2}}\| \leq \frac{K}{2^{n-1}}.$$

(Note that if we do not assume $X_0 = (1+a)/2$, but instead $X_0 = p(c)$ as we had earlier, then the same analysis shows that

$$|f_n(z)| \leq \frac{2|z|\kappa(t)^{2^{n-1}}}{1-\kappa(t)^{2^{n-1}}}$$

where now $t = \frac{2\operatorname{Re} q(z)}{1+|q(z)|^2}$, which is still in $[0, 1]$. However it may not be easy to dominate $|z|$ by a multiple of this t as we did before, unless $p(z)$ is of a very special form, like $(1+z)/2$.)

If $W(c) \subset S_\theta$ for $\theta < \pi/2$, set $z = te^{i\theta}$, and Equation (4.8) becomes

$$|f_n(z)| \leq \frac{2t\kappa(s)^{2^{n-1}}}{1-\kappa(s)^{2^{n-1}}} \leq \sec(\theta) \frac{s\kappa(s)^{2^{n-1}}}{1-\kappa(s)^{2^{n-1}}}(1+t^2) \leq \frac{\sec(\theta)(1+t^2)}{2^n} < \frac{C_\rho}{2^n},$$

for ϵ small enough, where $s = \frac{2t \cos \theta}{1+t^2}$ and C_ρ is any constant greater than $\sec(\frac{\rho}{2})$. \square

Remark 4.5. 1) With a little more work in the last proof one should be able to show that the maximum of $|f_n(z)|$ on $W(c)$, or on the intersection S_θ with the disk of radius $\|c\|$, is achieved at 0. This also seemed to be confirmed by numerical computations for various values of n . If this is the case then C_ρ may be replaced by 1 in the estimate in the last result. That is, $\|X_n - a^{\frac{1}{2}}\| \leq \frac{K}{2^n}$.

2) Thus $\|X_n\| \leq \|a^{\frac{1}{2}}\| + \frac{C}{2^{n-1}}$ for a constant C . One should also be able to get an estimate for $\|X_n^{-1}\|$. Indeed using Crouziex's functional calculus $\|X_n^{-1}\| \leq K\|g_n\|_{W(c)}$ where

$$g_n(z) = \frac{2}{1+z} (1+\kappa(z)^{2^n})^{-1} [(1+\kappa(z))(1+\kappa(z)^2) \cdots (1+\kappa(z)^{2^{n-1}})].$$

We expect that $\|g_n\|_{W(c)} = 2^n$ if a is not invertible (indeed in this case we have $\|g_n\|_{W(c)} \geq g_n(0) = 2^n$).

3) If a is accretive one may apply Newton's method to $a + \frac{1}{n}1$, to get approximants for $a^{\frac{1}{2}}$. This suggests at first sight that the following variant of Newton's method might work: $X_{n+1} = \frac{1}{2}(X_n + X_n^{-1}(a + \frac{1}{n}1))$. However since X_n^{-1} may be growing at an order of 2^n or faster this seems dangerous. We conjecture that $X_{n+1} = \frac{1}{2}(X_n + X_n^{-1}(a + \frac{1}{3^n}1))$ would work for all accretive operators a on a Hilbert space, and possibly also in a Banach algebra.

Proposition 4.6. *If x is a matrix with no strictly negative eigenvalues, and a square root in $\{x\}''$, then Newton's method with $x_0 = (x+1)/2$ converges to the principal square root $x^{\frac{1}{2}}$.*

Proof. If x is invertible then this follows from [20, Theorem 6.9]. By [20, Theorem 6.10] we just need to show that if 0 is an eigenvalue then it is a semisimple eigenvalue, that is, there is no nontrivial Jordan block for the eigenvalue 0. If there was such a nontrivial Jordan block J_0 then first suppose that $x = V^{-1}J_0V$. Then x has no square root as is well known (see e.g. [20, Exercise 1.25]). Otherwise, suppose that $x = V^{-1}(J_0 \oplus z)V$. if p is the support projection of J_0 then $V^{-1}pV$ commutes with x and hence also with $x^{\frac{1}{2}}$. Thus $V^{-1}pVx^{\frac{1}{2}}$ is a square root of $V^{-1}pVx = V^{-1}J_0V$. However J_0 has no square root as we said above, a contradiction. \square

5. THE GEOMETRIC MEAN, AND SOLVING $xa^{-1}x = b$

In this section we note that Drury's results from [16, Section 3] for the geometric mean of matrices with (strictly) positive definite real part, generalize to strictly accretive elements in a unital operator algebra. We also establish a few more aspects of this mean. We remark that the geometric mean of positive matrices and operators dates back to work of Pusz-Woronowicz and Ando (see [25] for a survey).

Theorem 5.1. *Let a and b be strictly accretive elements in a unital operator algebra. Then*

$$G = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}$$

is strictly accretive too. Moreover G is the unique strictly accretive solution to the equation $xa^{-1}x = b$, and $G = b^{\frac{1}{2}}(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{\frac{1}{2}}b^{\frac{1}{2}}$.

Proof. We slightly rewrite Drury's argument. The first part of the proof works in any unital Banach algebra: note that

$$t1 + a^{-\frac{1}{2}}ba^{-\frac{1}{2}} = a^{-\frac{1}{2}}(ta + b)a^{-\frac{1}{2}}, \quad t \geq 0,$$

is invertible since $ta + b$ is strictly accretive. So the spectrum of $a^{-\frac{1}{2}}ba^{-\frac{1}{2}}$ contains no negative numbers or 0, and by the spectral mapping theorem the spectrum of $(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}$ is contained in \mathbb{H} . Similarly for the spectrum of $(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{\frac{1}{2}}$. Clearly $Ga^{-1}G = b$. By Lemma 2.1 we have

$$\frac{2}{\pi} \int_0^\infty a^{-\frac{1}{2}}(t^21 + a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{-1}a^{-\frac{1}{2}}dt = a^{-\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{-\frac{1}{2}}a^{-\frac{1}{2}} = G^{-1}.$$

We may rewrite this (convergent) integral in the more symmetric form

$$\frac{2}{\pi} \int_0^\infty (ta + \frac{1}{t}b)^{-1} \frac{dt}{t}.$$

At this point we assume that A is an operator algebra. Note that for $0 < t < \infty$ we have that $ta + \frac{1}{t}b$ is strictly accretive, and so by Lemma 2.2, so is $(ta + \frac{1}{t}b)^{-1}$. By a basic fact about integrals of positive functions we see that the integral yields a strictly accretive element. By Lemma 2.2 the inverse G is strictly accretive too. Making the substitution $u = 1/t$ in the integral, we see that the symmetry is perfect, and so G equals

$$a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}} = b^{\frac{1}{2}}(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{\frac{1}{2}}b^{\frac{1}{2}}.$$

The argument in [16, Proposition 3.5] shows that there is a unique strictly accretive G satisfying $Ga^{-1}G = b$. There is one point in that proof where one needs that the spectrum of HG^{-1} contains no negative numbers, for H as in that paper, but this follows since

$$t1 + HG^{-1} = G^{-1}(tG + H), \quad t \geq 0,$$

is invertible since $tG + H$ is strictly accretive. \square

Drury writes G in the last result as $a\#b$, the *geometric mean*. As in [16, Proposition 3.1] we deduce:

Corollary 5.2 (Drury). *If a and b are as in the last result, and if $W(a)$ and $W(b)$ are inside S_θ for some $\theta < \frac{\pi}{2}$, then $W(a\#b) \subset S_\theta$.*

Lemma 5.3. *Let a and b be accretive operators on a Hilbert space H with a strictly accretive. Or, let a and b be accretive elements of a Banach algebra with a invertible and b strictly accretive. Then $a^{-\frac{1}{2}}ba^{-\frac{1}{2}}$ is of type M .*

Proof. If a is strictly accretive then there exists $\epsilon > 0$ with $a \geq \epsilon I$. We have

$$\|(a^{-\frac{1}{2}}ba^{-\frac{1}{2}} + t1)^{-1}\| \leq \|a^{\frac{1}{2}}\|^2 \|(b + ta)^{-1}\|.$$

For $\zeta \in H$ we have

$$t\epsilon\|\zeta\|^2 \leq t \operatorname{Re}\langle a\zeta, \zeta \rangle \leq \operatorname{Re}\langle (b + ta)\zeta, \zeta \rangle \leq \|(b + ta)\zeta\| \|\zeta\|.$$

Dividing by $\|\zeta\|$ and letting $\zeta = (b + ta)^{-1}\eta$ we obtain

$$\|(b + ta)^{-1}\eta\| \leq \frac{1}{t\epsilon} \|\eta\|, \quad \eta \in H.$$

It follows that

$$\|(a^{-\frac{1}{2}}ba^{-\frac{1}{2}} + t1)^{-1}\| \leq \|a^{\frac{1}{2}}\|^2 \frac{1}{t\epsilon}$$

so that $a^{-\frac{1}{2}}ba^{-\frac{1}{2}}$ is of type M .

If a and b are accretive elements of a Banach algebra with a invertible and b strictly accretive, and if $t \geq 0$, then

$$\operatorname{Sp}(ta + b) \subset W(ta + b) = \{\varphi(ta + b) : \varphi \in S(A)\} = \{t\varphi(a) + \varphi(b) : \varphi \in S(A)\} \subset \mathbb{H}.$$

Thus $ta + b$ is invertible, so that $t1 + a^{-\frac{1}{2}}ba^{-\frac{1}{2}}$ is invertible and $-t \notin \operatorname{Sp}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})$. Hence by the discussion above Lemma 2.4, $a^{-\frac{1}{2}}ba^{-\frac{1}{2}}$ is type M . \square

Remark 5.4. If a and b are strictly accretive it need not follow that $W(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})$ contains no negative numbers. For example, let $a^{-1} = b$ be the 2×2 matrix with rows $[1 \ 1]$ and $[-2 \ \frac{1}{3}]$.

We define $a\#b = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}$ if a is strictly accretive and b is accretive, or if a invertible and accretive and b is strictly accretive. If a and b are simply accretive, and a is invertible, then we define $a\#b = \lim_{\epsilon \rightarrow 0^+} a\#(b + \epsilon 1)$, as in the final assertion of the next result.

Corollary 5.5. *Let a and b be accretive elements in a unital operator algebra with a strictly accretive; or with a invertible and b strictly accretive. Then $a\#b$ is accretive too. Indeed its numerical range is again in S_θ if $W(a)$ and $W(b)$ are inside S_θ for some $\theta \leq \frac{\pi}{2}$. This is also true if a and b are simply accretive, and a is invertible, if we define $a\#b = \lim_{\epsilon \rightarrow 0^+} a\#(b + \epsilon 1)$. The latter limit exists and is accretive, and is a solution to the equation $za^{-1}z = b$.*

Proof. If a is strictly accretive apply the theorem (or its proof) with b replaced by $b + \epsilon$, and let $\epsilon \rightarrow 0^+$, using Lemmas 2.4 and 5.3. This allows one to see that

$$\|(a^{-\frac{1}{2}}(b + \epsilon 1)a^{-\frac{1}{2}})^{\frac{1}{2}} - (a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}\| \leq K\|\epsilon a^{-1}\|^{\frac{1}{2}} \rightarrow 0$$

as $\epsilon \rightarrow 0$. So $a\#(b + \epsilon 1) \rightarrow a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}$ as $\epsilon \rightarrow 0$. Since $W(b + \epsilon 1) \subset S_\theta$ if $W(b) \subset S_\theta$ the last assertion follows easily from Corollary 5.2.

Suppose that a is invertible and b strictly accretive. By the above $(a + \epsilon 1)\#b$ is accretive with numerical range in S_θ if $W(a)$ and $W(b)$ are inside S_θ . If $z_\epsilon = ((a + \epsilon 1)^{-\frac{1}{2}}b(a + \epsilon 1)^{-\frac{1}{2}})^{\frac{1}{2}}$ we have $(a + \epsilon 1)\#b - a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}$ equal to

$$((a + \epsilon 1)^{\frac{1}{2}} - a^{\frac{1}{2}})z_\epsilon(a + \epsilon 1)^{\frac{1}{2}} + a^{\frac{1}{2}}(z_\epsilon - z_0)(a + \epsilon 1)^{\frac{1}{2}} + a^{\frac{1}{2}}z_0((a + \epsilon 1)^{\frac{1}{2}} - a^{\frac{1}{2}}).$$

The last of these three terms has limit 0. The middle term has limit 0 too since by Lemmas 2.4 and 5.3,

$$\|z_\epsilon - z_0\| \leq K\|(a + \epsilon 1)^{-\frac{1}{2}}b(a + \epsilon 1)^{-\frac{1}{2}} - a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\|^{\frac{1}{2}} \rightarrow 0$$

as $\epsilon \rightarrow 0$. This also uses the fact that $(a + \epsilon 1)^{\frac{1}{2}} \rightarrow a^{\frac{1}{2}}$, and the continuity of the inverse. We deduce that $\|z_\epsilon\|$ is bounded independently of ϵ , and so the first of the three terms has limit 0 too. Thus $(a + \epsilon 1)\#b \rightarrow a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}$ as $\epsilon \rightarrow 0$. Hence the latter is accretive with numerical range in S_θ , since the former has these properties.

Finally, if a and b are simply accretive, and a is invertible, then $a^{-\frac{1}{2}}(b + \epsilon 1)a^{-\frac{1}{2}}$ is type M by Lemma 5.3, we have by Lemma 2.4 that

$$\|a\#(b + \epsilon_1 1) - a\#(b + \epsilon_2 1)\| \leq K\|a^{\frac{1}{2}}\|^2\|a^{-\frac{1}{2}}(\epsilon_1 - \epsilon_2)a^{-\frac{1}{2}}\|^{\frac{1}{2}}.$$

Thus $(a\#(b + \epsilon 1))$ is a Cauchy net, hence convergent to an accretive element z such that, as above, $W(z) \subset S_\theta$ if $W(a)$ and $W(b)$ are inside S_θ . It is an exercise that it is a solution to the equation $za^{-1}z = b$. \square

Remark 5.6. In the setting of Corollary 5.5, the same proof and Corollary 2.6 show that if in addition the arguments of numbers in $W(a)$ and $W(b)$ are inside $[\alpha, \beta]$ for $-\frac{\pi}{2} \leq \alpha \leq \beta \leq \frac{\pi}{2}$, then the same is true for $W(a\#b)$.

We also remark that solutions to $za^{-1}z = b$ are not unique without the earlier strictly accretive hypothesis. (for example let $a = \text{diag}(i, 2), b = \text{diag}(i, 1/2)$). However for any solution z to this equation, $a^{-\frac{1}{2}}za^{-\frac{1}{2}}$ is a square root of $a^{-\frac{1}{2}}ba^{-\frac{1}{2}}$.

Corollary 5.7. *Let a and b be accretive invertible elements in a unital operator algebra. Then with the definitions above, $a\#b = b\#a$.*

Proof. First assume that b is strictly accretive. By the last proof, we have

$$a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}} = \lim_{\epsilon \rightarrow 0^+} (a + \epsilon 1)\#b = \lim_{\epsilon \rightarrow 0^+} b\#(a + \epsilon 1) = b^{\frac{1}{2}}(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})^{\frac{1}{2}}b^{\frac{1}{2}},$$

the last equality by a fact in the proof of Corollary 5.5.

If b is not strictly accretive, we have by the above that $a\#b$ equals

$$\lim_{\epsilon \rightarrow 0^+} a^{\frac{1}{2}}(a^{-\frac{1}{2}}(b + \epsilon 1)a^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}} = \lim_{\epsilon \rightarrow 0^+} (b + \epsilon 1)^{\frac{1}{2}}((b + \epsilon 1)^{-\frac{1}{2}}a(b + \epsilon 1)^{-\frac{1}{2}})^{\frac{1}{2}}(b + \epsilon 1)^{\frac{1}{2}}.$$

This may be rewritten as $\lim_{\epsilon \rightarrow 0^+} b^{\frac{1}{2}}((b + \epsilon 1)^{-\frac{1}{2}}a(b + \epsilon 1)^{-\frac{1}{2}})^{\frac{1}{2}}b^{\frac{1}{2}}$. Now both $(b + \epsilon 1)^{-\frac{1}{2}}a(b + \epsilon 1)^{-\frac{1}{2}}$ and $b^{-\frac{1}{2}}(a + \epsilon 1)b^{-\frac{1}{2}}$ are type M by Lemma 5.3, so the norm of the difference of the square roots of these two products is dominated by Lemma 2.4 by

$$K\|(b + \epsilon 1)^{-\frac{1}{2}}a(b + \epsilon 1)^{-\frac{1}{2}} - b^{-\frac{1}{2}}(a + \epsilon 1)b^{-\frac{1}{2}}\|^{\frac{1}{2}} \rightarrow 0$$

as $\epsilon \rightarrow 0$. Thus we see

$$a\#b = \lim_{\epsilon \rightarrow 0^+} b^{\frac{1}{2}}(b^{-\frac{1}{2}}(a + \epsilon 1)b^{-\frac{1}{2}})^{\frac{1}{2}}b^{\frac{1}{2}} = b\#a$$

as desired. \square

We remark that in the case that both a and b are simply accretive and a is invertible, $a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}$ may not make sense. For example if $b = a^{-1} = i$ in \mathbb{C} then $a^{-\frac{1}{2}}ba^{-\frac{1}{2}} = b^2 = -1$, whose principal root is undefined.

Lemma 5.8. *If a and b are accretive elements in a unital operator algebra such that a and b commute, and if a is invertible, then $a\#b = a^{\frac{1}{2}}b^{\frac{1}{2}}$, and this is accretive.*

Proof. First suppose that b is also strictly accretive. Claim: $a^{-\frac{1}{2}}b^{\frac{1}{2}}$ has spectrum in \mathbb{H} . Indeed, if χ is a character of the unital Banach algebra generated by a and b then $\chi(a^{-\frac{1}{2}}b^{\frac{1}{2}}) = \chi(a)^{-\frac{1}{2}}\chi(b)^{\frac{1}{2}}$, which is in \mathbb{H} . By the Claim and the unicity of roots mentioned in the Introduction, we have

$$(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}} = (a^{-1}b)^{\frac{1}{2}} = a^{-\frac{1}{2}}b^{\frac{1}{2}}.$$

Hence $a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}$.

More generally, $a\#b = \lim_{\epsilon \rightarrow 0^+} a\#(b + \epsilon 1) = \lim_{\epsilon \rightarrow 0^+} a^{\frac{1}{2}}(b + \epsilon 1)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}$. \square

Similarly, if a and b are any accretive elements in a unital operator algebra such that a and b commute, then $\lim_{\epsilon \rightarrow 0^+} (a + \epsilon)\#b = a^{\frac{1}{2}}b^{\frac{1}{2}}$, and this is accretive. This shows that Corollary 5.5 is in some sense a noncommutative variant of the fact from [2] that $a^{\frac{1}{2}}b^{\frac{1}{2}}$ is accretive for accretive commuting elements in a unital operator algebra. We noted in [8, Example 3.13] that the latter fact is false in a Banach algebra. Hence none of the results above in this section are true for general Banach algebras.

It is easy to show that for positive scalars s, t we have $(sa)\#(tb) = \sqrt{st}(a\#b)$ for a, b accretive and a invertible.

Proposition 5.9. *Let a and b be accretive invertible elements in a unital operator algebra. Then with the definitions above, $(a\#b)^{-1} = a^{-1}\#b^{-1}$.*

Proof. By Lemma 2.2, a^{-1}, b^{-1} and $(b + \epsilon 1)^{-1}$ are accretive. Hence both $a^{\frac{1}{2}}(b + \epsilon 1)^{-1}a^{\frac{1}{2}}$ and $a^{\frac{1}{2}}(b^{-1} + \epsilon 1)a^{\frac{1}{2}}$ are type M by Lemma 5.3. By Lemma 2.4 the norm of the difference of the square roots of these two products is dominated by

$$K\|a^{\frac{1}{2}}(b + \epsilon 1)^{-1}a^{\frac{1}{2}} - a^{\frac{1}{2}}(b^{-1} + \epsilon 1)a^{\frac{1}{2}}\|^{\frac{1}{2}} \rightarrow 0$$

as $\epsilon \rightarrow 0$. Thus we see

$$a^{-1}\#b^{-1} = \lim_{\epsilon \rightarrow 0^+} a^{-\frac{1}{2}}(a^{\frac{1}{2}}(b + \epsilon 1)^{-1}a^{\frac{1}{2}})^{\frac{1}{2}}a^{-\frac{1}{2}}.$$

It is easy to see that the product of this and $a\#b = \lim_{\epsilon \rightarrow 0^+} a\#(b + \epsilon 1)$ is

$$\lim_{\epsilon \rightarrow 0^+} a^{-\frac{1}{2}}(a^{\frac{1}{2}}(b + \epsilon 1)^{-1}a^{\frac{1}{2}})^{\frac{1}{2}}a^{-\frac{1}{2}} \cdot a^{\frac{1}{2}}(a^{-\frac{1}{2}}(b + \epsilon 1)a^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}} = 1$$

using the fact mentioned in the introduction that $(x^{-1})^{\frac{1}{2}} = (x^{\frac{1}{2}})^{-1}$. Similarly, $(a\#b)(a^{-1}\#b^{-1}) = 1$ as desired. \square

Proposition 5.10. *Suppose that a, c are invertible in $B(H)$, and that a and b are accretive in $B(H)$. Then $c^*(a\#b)c = (c^*ac)\#(c^*bc)$. In particular $(c^*bc)^{\frac{1}{2}}$ equals $c^*((cc^*)^{-1}\#b)c$. Also, $(a + b)\#(a^{-1} + b^{-1})^{-1} = a\#b$ if a, b are strictly accretive.*

Proof. First assume that a, b are strictly accretive. Then $a\#b$ is strictly accretive by Theorem 5.1. Also, if c is invertible, then c^*ac and c^*bc are strictly accretive (for example, if $a \geq \epsilon 1$ then $c^*ac \geq \epsilon c^*c$, and the latter is strictly positive. Hence

$(c^*ac)\#(c^*bc)$ is strictly accretive, and its inverse, by the formula in the proof of Theorem 5.1 is

$$\frac{2}{\pi} \int_0^\infty (c^*(ta + \frac{1}{t}b)c)^{-1} \frac{dt}{t} = c^{-1}G^{-1}(c^{-1})^*,$$

where $G = a\#b$. so that $c^*Gc = (c^*ac)\#(c^*bc)$.

If b is strictly accretive and a is invertible, then $c^*((a + \epsilon 1)\#b)c = (c^*(a + \epsilon 1)c)\#(c^*bc)$. Clearly the left side converges to $c^*(a\#b)c$ as $\epsilon \rightarrow 0^+$. The right side converges to $(c^*ac)\#(c^*bc)$ again by a slight variant of the proof of Corollaries 5.5, 5.7, or Proposition 5.9.

If a, b are merely accretive, and a is invertible, then we have from the above that

$$c^*(a\#(b + \epsilon 1))c = (c^*ac)\#(c^*(b + \epsilon 1)c).$$

The left side converges to $c^*(a\#b)c$ again. If $w = c^*ac$ then $w^{-\frac{1}{2}}(c^*(b + \epsilon 1)c)w^{-\frac{1}{2}}$ and $w^{-\frac{1}{2}}(c^*bc + \epsilon 1)w^{-\frac{1}{2}}$ are both type M . So by a variant of the last part of the proof of Corollary 5.7 we have

$$c^*(a\#b)c = \lim_{\epsilon \rightarrow 0^+} w^{\frac{1}{2}}(w^{-\frac{1}{2}}(c^*bc + \epsilon 1)w^{-\frac{1}{2}})^{\frac{1}{2}}w^{-\frac{1}{2}} = (c^*ac)\#(c^*bc).$$

If a, b are strictly accretive then a^{-1} , b^{-1} , $(a + b)$ and $a^{-1} + b^{-1}$ are also strictly real positive by Lemma 2.2. Then the result follows as in the literature from the unicity of the solution to $xa^{-1}x = b$, and the relations $(a\#b)a^{-1}(a\#b) = b$ and $(a\#b)b^{-1}(a\#b) = a$. \square

Probably with slightly more work one can generalize the second assertion of the last result to the case that $a, b, a + b$ and $a^{-1} + b^{-1}$ are accretive and invertible. Also, it seems possible that many of our results on the geometric mean can be extended to the case when both operators are accretive with no further conditions. We remark also that the well known algebraic-geometric-harmonic mean inequality for the geometric mean of positive matrices fails badly for accretive matrices, as may be seen by considering $a = \text{diag}(1 + i, 1 - i)$, $b = \text{diag}(1 - i, 1 + i)$ (here all three means are positive matrices, but violate—actually reverse—the usual inequalities).

6. THE BINOMIAL AND VISSER METHODS FOR THE SQUARE ROOT

We expect that the ‘binomial method’

$$(6.1) \quad X_{n+1} = \frac{1}{2}(b + X_n^2), \quad X_0 = 0,$$

and its variant the ‘Visser method’

$$X_{n+1} = X_n + \alpha(a - X_n^2), \quad X_0 = \frac{1}{2\alpha}I,$$

work in Banach algebras, under reasonable hypotheses. Here $b = 1 - a$, and it is expected that these schemes converge to $1 - a^{\frac{1}{2}}$ and $a^{\frac{1}{2}}$ respectively. Of course it is well known that if $\|1 - a\| \leq 1$ then the binomial series for $(1 - (1 - a))^t$ converges to a^t (see e.g. [8, Proposition 3.3]; this may have been first done in [23]). However the binomial series is a little different from the binomial method above. For operators a on a Hilbert space one can (more or less easily, depending on the numerical range concerned) prove convergence results for the binomial method using the disk algebra functional calculus (coming from von Neumann’s inequality) or more generally Crouzeix’s functional calculus mentioned at the end of the introduction, which essentially reduces the computation to one about scalars. Then the matching Visser

method result follows by the usual substitution turning the binomial method into the Visser method (see the proof on [20, p. 159], or Corollary 6.3 below). This provides an effective iterative ‘polynomial approximation’ for the square root of any operator in \mathfrak{F}_A for an operator algebra A . The following is intimately connected with the complex dynamics of the Mandelbrot set. Indeed the scalar case of the ‘binomial method’ (6.1) if we change variables $w = 2x$, and let $c = b/2$, becomes the usual quadratic iteration $w_{n+1} = w_n^2 + c$ used to define the Mandelbrot set. The ‘main cardioid’ for the binomial method is the set of attracting fixed points of $z \mapsto \frac{1}{2}(z^2 + b)$; and this may be obtained almost identically to the Mandelbrot set case [3, p. 15] from the open unit disk $D(0, 1)$ by subtracting the latter from 1, then squaring all elements, and then subtracting the resulting set from 1.

Theorem 6.1. *Let b be a Hilbert space operator with numerical range contained in a compact subset E of the cardioid $2z - z^2$ for $|z| < 1$, or more generally contained in the union of E and the closed unit disk $\bar{D}(0, 1)$. The binomial method (6.1) applied to b converges to $1 - a^{\frac{1}{2}}$, where $a = 1 - b$. As a special case, for any contraction b on a Hilbert space (that is, $\|b\| \leq 1$), the binomial method converges to $1 - a^{\frac{1}{2}}$.*

Proof. Let \mathcal{D} be the union of the indicated disk and cardioid. Define polynomials $q_n(z)$ on \mathcal{D} by $q_0 = 0$ and $q_{n+1}(z) = \frac{1}{2}(z + q_n(z)^2)$. Then $X_n = q_n(b)$, and we need to show that $\|X_n - 1 + c\| \rightarrow 0$, where $c = (1 - b)^{\frac{1}{2}}$. By the scalar case (see e.g. [20, Theorem 6.14]), $(q_n(z))$ converges pointwise on the interior of the cardioid to $1 - (1 - z)^{\frac{1}{2}}$, and the latter function is certainly analytic on some open subset of \mathcal{D} . Moreover (q_n) is well known (and easily seen) to be uniformly bounded on the ‘main cardioid’. (For example, this cardioid is bounded by 4, and so if $z_n = q_n(z)$ for some z in the cardioid, and if $|z_n| = 4 + a$, where $a > 0$ then

$$|z_{n+1}| \geq \frac{1}{2}|z_n|^2 - 2 = \frac{1}{2}(4 + a)^2 - 2 = 6 + 4a + \frac{1}{2}a^2 > 4 + 4a.$$

By induction $|z_{n+k}| > 4 + 4^k a \rightarrow \infty$ as $k \rightarrow \infty$, so $z_n \rightarrow \infty$, which contradicts one of the definitions of the Mandelbrot set—as the points whose iterations are bounded.) Thus by Vitali’s theorem combined with Montel’s theorem (see [3, Section 3.3]), (q_n) converges uniformly on any compact subset of the interior of the cardioid. We next show that $r_n(z) = q_n(z) - 1 + (1 - z)^{\frac{1}{2}} \rightarrow 0$ uniformly on the disk. We use an idea in the argument for the scalar case from [20]. Let $w = (1 - z)^{\frac{1}{2}}$. We have

$$r_{n+1}(z) = \frac{1}{2}((z + q_n(z)^2) - 1 + w) = \frac{1}{2}(q_n(z) + 1 - w)r_n(z).$$

It is clear by induction that if $|z| \leq 1$ then $|q_n(z)| \leq 1$ for all n (indeed if this is true for n then by the binomial theorem we have $|q_{n+1}(z)| \leq \frac{1}{2}(1 + |q_n(z)|^2) \leq 1$). For $|z| \leq 1$ we have

$$|1 - w| = |1 - (1 - z)^{\frac{1}{2}}| \leq -\sum_{k=1}^{\infty} (-1)^k \binom{1/2}{k} = 1.$$

Hence

$$|r_{n+1}(z)| \leq \frac{1}{2}(|q_n(z)| + |1 - w|)|r_n(z)| \leq |r_n(z)|.$$

Thus $(|r_n(z)|)$ is decreasing with pointwise limit 0, so by Dini’s theorem (r_n) converges uniformly on the unit disk.

Let E be the (closed) numerical range of b . By the hypotheses on E , and the facts just established, (q_n) converges uniformly on E . By Crouzeix's functional calculus (or we could use the disk algebra functional calculus coming from von Neumann's inequality if b is a contraction), for some constant K we have

$$\|q_m(a) - q_n(a)\| \leq K\|q_m - q_n\|_E, \quad m, n \in \mathbb{N}.$$

Thus (X_n) is Cauchy, and hence convergent to w say. We have $w = \frac{1}{2}(w^2 + b)$, so $a = (1 - w)^2$. We also note that any point in the spectrum of w is a limit of $(q_k(z))$ for some $z \in E$, and hence equals $1 - (1 - z)^{\frac{1}{2}}$. Thus the spectrum of $1 - w$ is the right half plane, and hence $1 - w$ is the principal square root of a . \square

Remark 6.2. 1) If b in the last proof is a contraction then the part of the proof using Dini's theorem gives a seemingly more controlled convergence, with the 'error term' dominated by a decreasing null sequence.

2) One may rephrase the last result in terms of subsets of the unit disk, instead of subsets of the cardioid. Indeed the homeomorphism between that disk and the cardioid mentioned before the theorem statement gives a kind of passage between statements about b and statements about $1 - a^{\frac{1}{2}} = 1 - (1 - b)^{\frac{1}{2}}$.

Corollary 6.3. *Let a be an operator on a Hilbert space with the numerical range of $1 - t^2a$ (that is, $1 - t^2W(a)$) contained in $E \cup \bar{D}(0, 1)$, where E is as in the last theorem. Then the Visser method $X_{k+1} = X_k + \frac{t}{2}(a - X_k)^2$ with initial guess $X_0 = \frac{1}{t}I$, converges to $a^{\frac{1}{2}}$. In particular this holds if $a \in \mathfrak{c}_{B(H)}$.*

Proof. By Theorem 6.1 the binomial method applied to $b = 1 - t^2a$ gives a sequence (B_n) converging to $1 - ta^{\frac{1}{2}}$. So $\frac{1}{t}(1 - B_n) \rightarrow a^{\frac{1}{2}}$. However one can check that $\frac{1}{t}(1 - B_n)$ coincides with the n th step in the Visser method in the statement.

If $a \in \mathfrak{c}_{B(H)}$ then $\|1 - t^2a\| \leq 1$ for some $t > 0$, so we are in the special case that b is a contraction in the last theorem. \square

Remark 6.4. There is probably a similar method for the p th root, and results similar to the two above in that case.

7. NEWTON'S METHOD FOR THE P TH ROOT

Newton's method for the p th root of a , for $p > 1$, is

$$X_{k+1} = \frac{1}{p}X_k((p-1)I + X_k^{-p}a).$$

With $X_0 = I$ or $X_0 = \frac{1}{2}(a + I)$ this method need not work for accretive matrices. Indeed it fails even for some scalars in the right half plane (see the discussion on page 178–179 of [20]). In the light of the scalar case, one would expect that Newton's method for the p th root of a with starting guess $X_0 = I$ works with some restriction on a , such as that the numerical range of a should be in the region of convergence for the scalar case. Let

$$\mathcal{D} = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } |z| \leq 1 + \epsilon\} \cup \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } |z - 1| \leq 1\}.$$

Proposition 7.1. *Let $p > 1$ be an integer. There exists $\epsilon > 0$ such that for any Hilbert space operator a with numerical range contained in the set \mathcal{D} above, Newton's method for the p th root above, with initial point $X_0 = I$, converges to $a^{\frac{1}{p}}$.*

Proof. Define a sequence of rational functions

$$q_{k+1}(z) = \frac{1}{p} q_k(z) \left((p-1) + \frac{z}{q_k(z)^p} \right), \quad q_0 = 1,$$

for all z where this makes sense (that is, where $q_k(z)$ is defined for all $k \in \mathbb{N}$). By [21, Lemma 2.11] there exists $\epsilon > 0$ such the sequence above does make sense if $\operatorname{Re}(z) > 0$ and $|z| \leq 1 + \epsilon$, and the (q_k) converges to $z^{\frac{1}{p}}$ uniformly on any compact subset of $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } |z| \leq 1 + \epsilon\}$.

We next consider the set $K_1 = \{z \in \bar{D}(1, 1) : \operatorname{Re} z > \frac{1}{4}\}$. On this set, if $(c_k)_{k \geq 2}$ is the sequence of positive numbers with sum 1 from Lemma 1 in [17, Section 3], we have $|c_2 + c_3(1 - z)| \leq d < c_2 + c_3$ for some constant d . This is because

$$(c_2 + c_3)^2 - |c_2 + c_3(1 - z)|^2 = 2c_2c_3\operatorname{Re}(z) + c_3^2(1 - |1 - z|^2) \geq \frac{1}{2}c_2c_3.$$

By the argument in the just mentioned lemma from [17] we have $|1 - z/q_1(z)^p| \leq \alpha$ for all $z \in K_1$, where $\alpha = d + \sum_{k=4}^{\infty} c_k < 1$, and

$$|1 - z/q_n(z)^p| \leq |1 - z/q_1(z)^p|^{2^{n-1}} \leq \alpha^{2^{n-1}}.$$

The sequence $(q_n(z))$ is well defined on K_1 by the argument in [17]. And $(q_n(z)^p)$ and therefore also $(q_n(z))$ is uniformly bounded on K_1 , by a constant M say, since $q_n(z)^p = \frac{z}{1 - (1 - z/q_n(z)^p)}$. Moreover, since

$$|q_{n+1}(z) - q_n(z)| = \frac{1}{p} |q_n(z)| |1 - z/q_n(z)^p| \leq \frac{M}{p} \alpha^{2^{n-1}}, \quad z \in K_1,$$

it is easy to see that (q_n) is uniformly Cauchy on K_1 , so uniformly convergent. Thus the rational functions (q_k) converge to $z^{\frac{1}{p}}$ uniformly on any compact subset of the set \mathcal{D} above.

Suppose that $W(a) \subset \mathcal{D}$. By Crouzeix's functional calculus, for some constant K we have

$$\|q_m(a) - q_n(a)\| \leq K \|q_m - q_n\|_{W(a)}, \quad m, n \in \mathbb{N}.$$

Thus (X_n) is Cauchy, and hence convergent to w say. Since

$$pX_k^{p-1}(X_{k+1} - X_k + \frac{1}{p}X_k) = a,$$

in the limit we have $w^p = a$. We also note that by spectral theory any point in the spectrum of w is a limit of $(q_k(z))$ for some $z \in E$ (namely $z = \chi(a)$ where χ is a character of the closed algebra generated by 1 and a), and hence equals $z^{\frac{1}{p}} \in S_{\frac{\pi}{2p}}$. Thus w is the principal p th root of a . \square

Remark 7.2. (1) It is easy to see, by the same argument as the scalar case (without needing Crouzeix's functional calculus), that the convergence in the last theorem is 'quadratic' in the sense of the usual Newton p th root convergence for scalars.

(2) Experimentation shows that the polynomials (q_n) in the last proof seem to converge uniformly on the set $\bar{\mathcal{D}}$. If this is indeed the case then the last proof shows that Newton's method for the p th root above converges to $a^{\frac{1}{p}}$ for any Hilbert space operator a with numerical range contained in the set $\bar{\mathcal{D}}$.

(3) A similar idea of course shows that Newton's method for the p th root converges for Hilbert space operators with $T \geq 0$, no doubt a well known fact. Indeed the Newton iterates take place in the unital C^* -algebra generated by T ,

which by Gelfand theory may be taken to be $C(E)$ for a compact set $E \subset [0, \infty)$. The functions (q_n) in the last proof are easily seen to be decreasing (certainly for $n \geq 2$) and hence converge uniformly on E by Dini's theorem. Hence the Newton iterates are $\|q_n(T) - T^{\frac{1}{p}}\| = \|q_n - q\|_E \rightarrow 0$, where $q(t) = t^{\frac{1}{p}}$.

Proposition 7.3. *Let a be an element in a unital Banach algebra A with $\|1 - a\| < 1$. Let $p > 1$ be an integer. Then Newton's method for the p th root above, with initial point $X_0 = I$, converges to $a^{\frac{1}{p}}$ (indeed there exist constants $\delta < 1$ and C with $\|X_k - a^{\frac{1}{p}}\| \leq C\delta^{2^k}$ for each k , and the latter has quadratic convergence). In particular, this is the case by Proposition 2.8 if a is strictly accretive and $\|1 - 2a\| \leq 1$.*

Proof. We follow the argument in [17], noting that Lemma 1 there holds with the same proof to show that the Newton sequence is well defined, and if $\delta = \|1 - aX_1^{-p}\|$ then

$$\|1 - aX_k^{-p}\| \leq \|(1 - aX_1^{-p})^{2^{k-1}}\| \leq \|1 - aX_1^{-p}\|^{2^{k-1}} = \delta^{2^{k-1}},$$

and

$$\delta = \|1 - aX_1^{-p}\| = \left\| \sum_{i=2}^{\infty} c_i (1-a)^i \right\| < \sum_{i=2}^{\infty} c_i = 1.$$

So $aX_k^{-p} \rightarrow 1$ rapidly with k (the 'error' is dominated by $\delta^{2^{k-1}}$ where $\delta < 1$, which has quadratic convergence). Hence $X_k^p a^{-1} \rightarrow 1$ and $X_k^p \rightarrow a$, which means that $\|X_k^p - 1\| = \|X_k^p - a + a - 1\| < 1$ for k large. Hence $\|X_k - 1\| < 1$ by e.g. [8, Proposition 3.3] and its proof, so that (X_k) is bounded. It follows as in the proof of [17, Theorem 5] (which is a result about the scalar case, not operators)

$$\|X_{k+1} - X_k\| = \frac{1}{p} \|X_k(1 - aX_k^{-p})\| \leq \frac{K}{p} \|1 - aX_1^{-p}\|^{2^{k-1}} \leq \frac{K}{p} \delta^{2^{k-1}}$$

for a constant K . Hence (X_n) is Cauchy, and we can finish the proof as in Proposition 7.1. By the usual triangle inequality argument we get $\|X_{k+m} - X_k\|$ and hence $\|X_k - a^{\frac{1}{p}}\|$, dominated by $C\delta^{2^k}$, for a constant C . \square

An important remark about other 'starting values' in Newton's method: If a is an element of a unital Banach algebra, and if Newton's method for the p th root of a starting at $X_0 = 1$ converges to $a^{1/p}$, then for any θ with $|\theta| \leq \pi$ say, Newton's method for the p th root of $e^{i\theta}a$ starting at $X_0 = e^{i\theta/p}$, converges. Its limit is $e^{i\theta/p}a^{1/p}$, which is a p th root of $e^{i\theta}a$. Indeed the latter is the principal p th root of $e^{i\theta}a$ by Theorem 2.5, if $W(e^{i\rho}a)$ contains no strictly negative numbers for all ρ between 0 and θ (including 0 and θ). To see this define $Y_k = e^{i\frac{\theta}{p}}X_k$. It is easy to see that (Y_k) coincides with the iterates in Newton's method for $e^{i\theta}a$ starting at $X_0 = e^{i\theta/p}$.

As in Iannazzo's paper [21] note that for any strictly accretive Hilbert space operator a , $b = a^{\frac{1}{2}}/\|a^{\frac{1}{2}}\|$ is also strictly accretive by e.g. a result on p. 181 of [19], and has norm ≤ 1 . So $W(b)$ lies in the set \mathcal{D} considered in Proposition 7.1 above. Thus Proposition 7.1 applies to b , and so we can use Newton's method to find $b^{\frac{1}{p}}$, from which $a^{\frac{1}{p}}$ is easily recovered.

Another method to find the p th root of a is to use the sign function studied in Section 3, in the way indicated in [5, Section 3] in the matrix case. In fact the beautiful arguments of [5, Section 3] go through with 'eigenvalues' replaced by

‘spectrum’. As in that reference, if p is odd we replace it by $2p$ and replace a by a^2 . If p is an integer multiple of 4 we keep dividing it by 2 and replacing a by the square root of a , until $p/2$ is odd. We then set

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a & 0 & 0 & \cdots & 0 \end{bmatrix},$$

(with the new a , if we had to change a as above). Then $a^{\frac{1}{p}}$ can be read off from the 1-2 entry of $\text{sign}(c)$. And in Section 3 we discussed the Newton method for $\text{sign}(c)$ and its convergence. We obtain, as in [5, Section 3]:

Theorem 7.4. *Suppose that a is an invertible element of a unital operator algebra with no negative numbers in its spectrum, and let v be the 1-2 entry of $\text{sign}(C)$ where C is as above. Then $a^{\frac{1}{p}} = \frac{p}{2\sigma}v$ where $\sigma = 1 + 2 \sum_{k=1}^r \cos(\frac{2\pi k}{p})$, and r is the greatest integer less than or equal to $p/4$.*

Halley’s method for the p th root of a , for $p > 1$, is

$$X_{k+1} = X_k \frac{(p-1)X_k^p + (p+1)a}{(p+1)X_k^p + (p-1)a},$$

with $X_0 = I$.

In the light of the scalar case [22], one would expect that this method should converge if the numerical range of a is inside the region of convergence for the scalar case, which is a much nicer region than that of the Newton method. This is the case:

Proposition 7.5. *Let $p > 1$ be an integer. For any strictly accretive element a , Halley’s method for the p th root above with initial point $X_0 = I$, converges to $a^{\frac{1}{p}}$.*

Proof. Define a sequence of rational functions

$$q_{k+1}(z) = q_k(z) \frac{(p-1)q_k(z)^p + (p+1)z}{(p+1)q_k(z)^p + (p-1)z}, \quad q_0 = 1,$$

for all z where this makes sense (that is, where $q_k(z)$ is well defined for all $k \in \mathbb{N}$). By [22, Corollary 5.3], $q_k(z)$ is well defined for all $k \in \mathbb{N}$ if $z \in \mathbb{H}$, and $q_k(z) \rightarrow z^{\frac{1}{p}}$ on \mathbb{H} . If $w_0(z) = z^{-\frac{1}{p}}$ and $w_k(z) = q_k(z) \cdot z^{-\frac{1}{p}}$, then

$$w_{k+1}(z) = w_k(z) \frac{(p-1)w_k(z)^p + (p+1)}{(p+1)w_k(z)^p + (p-1)},$$

for any $z \in \mathbb{H}$. By [22, Lemma 5.2] we have $|\arg(w_{k+1}(z))| \leq |\arg(w_k(z))|$, thus $|\arg(w_{k+1}(z))| < \frac{\pi}{2p}$ for any $z \in \mathbb{H}$. Therefore, $|\arg(q_{k+1}(z))| < \frac{\pi}{p}$, which means $q_k(z) : \mathbb{H} \rightarrow \mathbb{H}$ for any $k \in \mathbb{N}$. By Montel’s normality criterion (see [3, Section 3.3]), $(q_k(z))$ is normal. By Vitali’s Theorem, $(q_k(z))_{k=1}^\infty$ converges locally uniformly to $z^{\frac{1}{p}}$ on \mathbb{H} .

Suppose that $W(a) \subset \mathbb{H}$. By Crouzeix’s functional calculus, for some constant K we have

$$\|q_m(a) - q_n(a)\| \leq K \|q_m - q_n\|_{W(a)}, \quad m, n \in \mathbb{N}.$$

Thus (X_n) is Cauchy, and hence convergent to w say. If χ is a character of the closed algebra generated by 1 and a then

$$\chi(w) = \lim_k \chi(X_k) = \lim_k q_k(\chi(a)) = \chi(a)^{1/p} \neq 0.$$

Thus w is invertible. Since

$$(p+1)X_{k+1}X_k^p - (p-1)X_k^{p+1} = ((p+1)X_k - (p-1)X_{k+1})a,$$

in the limit we have $w^p = a$. Now we may finish as in Proposition 7.1. \square

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